

# Growth of matrix products and mixing properties of the horocycle flow

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## 1 Introduction

In this paper we investigate the following problem. Let  $H(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$  and let  $\Phi_* = \{\Phi_n\}$  be an arbitrary sequence of matrices from  $SL(2, \mathbb{R})$ . We will consider the sequence of products  $P_n(t) = \Phi_n H(t) \Phi_{n-1} H(t) \dots \Phi_1 H(t)$  and denote by  $\mathfrak{B}(\Phi_*)$  the set of those periods  $t \in \mathbb{R}_+$  for which the sequence  $\{P_n(t)\}$  is bounded:

$$\mathfrak{B}(\Phi_*) = \left\{ t \in \mathbb{R}_+ : \sup_{n \geq 1} \|P_n(t)\| < \infty \right\}.$$

The question is: *how large the set  $\mathfrak{B}(\Phi_*)$  can be?* We present three results on this subject. The first one shows that for every  $\{\Phi_n\}$  the set  $\mathfrak{B}(\Phi_*)$  is not “very large”:

**Theorem 1.** *For every sequence  $\Phi_*$ , the set  $\mathfrak{B}(\Phi_*)$  has finite measure.*

It should be noted that for sequences  $\Phi_*$  of some special types this was already established in [1]. Our main innovation, which gives us the possibility to handle the general case, is using of potential theory (Lemma 5).

The next two results demonstrate that the conclusion of Theorem 1 cannot be strengthened too much. Namely, Theorem 2 (section 5) shows that the exceptional set  $\mathfrak{B}(\Phi_*)$  can contain an arbitrary given sequence. In Theorem 3 (section 6) we produce an example of a sequence  $\Phi_*$  for which the set  $\mathfrak{B}(\Phi_*)$  is essentially unbounded, that is  $|\mathfrak{B}(\Phi_*) \cap [a, +\infty)| > 0$  for all  $a > 0$  (We denote by  $|E|$  the Lebesgue measure of a set  $E \subset \mathbb{R}$ ).

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**Motivation: stable quasi-mixing of the horocycle flow.** In a recent work of L. Polterovich and Z. Rudnick [1] the authors considered the behavior of one-parameter subgroup of a Lie group under the influence of a sequence of kicks. We remind some basic concepts of their paper.

Let a Lie group  $G$  act on a set  $X$ , and  $(h^t)_{t \in \mathbb{R}}$  be a one-parameter subgroup of  $G$ ; we consider it as a dynamical system acting on  $X$  with continuous time  $t$ . We perturb this system by a sequence of kicks  $\{\phi_i\} \subset G$ . The kicks arrive periodically in time with some positive period  $t$ . The dynamics of the kicked system is described by a sequence of products  $P_i(t) = \phi_i h^t \phi_{i-1} h^t \dots \phi_1 h^t$  that depend on the period  $t$ . We treat  $t$  as a parameter and  $i$  as a discrete time. Then the trajectory of a point  $x \in X$  is defined as  $x_i = P_i(t)x$ .

A dynamical property of a subgroup  $(h^t)$  is called *kick stable*, if for every sequence of kicks  $\{\phi_i\}$ , the kicked sequence  $P_i(t)$  inherits this property for a “large” set of periods  $t$ . The property we will concentrate on in this paper, is quasi-mixing.

A sequence  $\{P_i\}$  acting on a compact measure space  $(X, \mu)$  by measure-preserving automorphisms is called *mixing* if for any two  $L_2$ -functions  $F_1$  and  $F_2$  on  $X$

$$\int_X F_1(P_i x) F_2(x) d\mu \rightarrow \int_X F_1(x) d\mu \int_X F_2(x) d\mu$$

when  $i \rightarrow \infty$ . A sequence  $\{P_i\}$  is called *quasi-mixing* if there exists a subsequence  $\{i_k\} \rightarrow \infty$  such that for any two  $L_2$ -functions  $F_1$  and  $F_2$  on  $X$

$$\int_X F_1(P_{i_k} x) F_2(x) d\mu \rightarrow \int_X F_1(x) d\mu \int_X F_2(x) d\mu$$

when  $k \rightarrow \infty$ .

In what follows,  $G = PSL(2, \mathbb{R})$ ,  $\Gamma \subset PSL(2, \mathbb{R})$  is a lattice, that is a discrete subgroup such that the Haar measure of the quotient space  $X = PSL(2, \mathbb{R})/\Gamma$  is finite. The group  $PSL(2, \mathbb{R})$  acts on  $X$  by left multiplication. This action evidently preserves the Haar measure. The principal tool used in [1] for the study of stable mixing in this setting, is the Howe-Moore theorem which gives the geometric description of mixing systems: if the sequence  $\{P_i\}$  tends to infinity<sup>1</sup> then it is mixing. It was also shown that the converse is true. In a similar way, the quasi-mixing is equivalent to the unboundedness of the sequence  $\{P_i\}$ .

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<sup>1</sup>i.e., for every compact subset  $Q \subset G$  the sequence  $\{P_i\}$  eventually leaves  $Q$

It follows from the Howe-Moore theorem that the horocycle flow

$$H(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

on  $PSL(2, \mathbb{R})/\Gamma$  is mixing. An example given in [1, Remark 3.3.E] shows that this flow is not stably mixing. Our Theorem 1 says that *it is stably quasi-mixing*. This answers the question raised by Polterovich and Rudnick [1, Question 3.3.B].

Let us mention a corollary to Theorem 1 that pertains to second order difference equations. It was shown in [1] that for a kick sequence of the form  $\begin{pmatrix} 1 & 0 \\ c_n & 1 \end{pmatrix}$ , the unboundedness of the evolution is equivalent to the existence of unbounded solutions for the discrete Schrödinger-type equation

$$q_{k+1} - (2 + tc_k)q_k + q_{k-1} = 0, \quad k \geq 1. \quad (1)$$

So our result implies

**Corollary 1.** *For every sequence  $\{c_n\}$ , the set of the parameters  $t \in \mathbb{R}_+$  for which all solutions of the difference equation (1) are bounded, has finite measure.*

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## 2 Outline of the proof of Theorem 1

Our proof of Theorem 1 consists of several steps and uses some preliminary results (Lemmas 1-5 below). For convenience of a reader we begin with an outline of this proof.

**Step 1.** First of all we show, that the problem can be reduced to the case of bounded sequences of kicks  $\Phi_*$  (Lemma 1).

**Step 2.** For bounded sequences we use the Iwasawa's decomposition of  $2 \times 2$  matrices:

$$\Phi_n = \begin{pmatrix} 1 & s_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_n & 0 \\ 0 & \frac{1}{\lambda_n} \end{pmatrix} \begin{pmatrix} \cos \alpha_n & -\sin \alpha_n \\ \sin \alpha_n & \cos \alpha_n \end{pmatrix} := H(s_n)D(\lambda_n)R(\alpha_n).$$

with bounded sequences  $\{H(s_n)\}$  and  $\{D(\lambda_n)\}$  and  $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ . Denoting by

$$q = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n |\alpha_j|$$

we then consider two cases separately:  $q = 0$  and  $q > 0$ .

**Step 3.** In the case of “small” angles ( $q = 0$ ), the sequence  $\{\Phi_n\}$  is “close” (in some sense which we define below) to the bounded sequence  $\{H(s_n)D(\lambda_n)\}$  of upper-triangular matrices. This implies that the set  $\mathfrak{B}(\Phi_*)$  is bounded (see Lemma 3).

**Step 4.** In the case  $q > 0$  we extend our problem to the complex plane and consider  $SL(2, \mathbb{C})$ -matrices  $H(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ . Respectively,  $P_n(z) = \Phi_n H(z) \cdot \dots \cdot \Phi_1 H(z)$ . We show that the set

$$E = \{z \in \mathbb{C} : \limsup_{n \rightarrow \infty} \frac{\log \|P_n(z)\|}{n} = 0\}$$

is contained in  $\mathbb{R}$  and has finite length. So we not only prove that the sequence  $\{P_n\}$  is unbounded but prove that it has exponential growth for all  $t$  apart of a set of finite measure.

In order to show that

$$\limsup_{n \rightarrow \infty} \frac{\log \|P_n(z)\|}{n} > 0 \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (2)$$

we have to estimate  $\|P_n(z)\|$  from below. To this aim we use the quadratic form

$$Q(x) = \operatorname{Im}(x_1 \bar{x}_2), \quad x = (x_1, x_2) \in \mathbb{C}^2,$$

which has the following properties:

- (i) for arbitrary  $y$ ,  $\|y\|^2 \geq 2Q(y)$ ,
- (ii) for every  $z \in \mathbb{C}$  with  $\operatorname{Im} z > 0$ , one has

$$Q(H(z)\Phi_n H(z)x) \geq Q(x) \left(1 + \frac{|\alpha_n| \operatorname{Im} z}{2k(1 + |z|)}\right). \quad (3)$$

Due to these properties we get that

$$\overline{\lim}_{n \rightarrow \infty} \frac{\log \|P_n(z)\|}{n} = \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \left\| \prod_{1 \leq j \leq n}^{\curvearrowright} H(z/2) \Phi_j H(z/2) \right\| > 0.$$

(We denote by  $\prod_{1 \leq j \leq n}^{\curvearrowright} A_j$  the matrix product  $A_n A_{n-1} \dots A_1$ .)

The claim that  $|E| < \infty$  follows now from a potential theory lemma (Lemma 5) applied to the subharmonic functions

$$u_n(z) = \frac{\log \|P_n(z)\|}{n}.$$

### 3 Preliminaries

**Lemma 1.** *If the sequence of kicks  $\Phi_*$  is unbounded then the set  $\mathfrak{B}(\Phi_*)$  is empty.*

*Proof.* Aiming at a contradiction, we assume that for some  $t > 0$  the sequence  $\{\|P_n(t)\|\}$  is bounded by  $M$ . Taking into account that  $\|A^{-1}\| = \|A\|$  for  $A \in SL(2, \mathbb{R})$ , we obtain

$$\begin{aligned} \|\Phi_n\| &= \|P_n(t) (P_{n-1}(t))^{-1} (H(t))^{-1}\| \\ &\leq \|P_n(t)\| \cdot \|(P_{n-1}(t))^{-1}\| \cdot \|(H(t))^{-1}\| \leq M^2 \|H(t)\| \end{aligned}$$

which contradicts to the unboundedness of  $\{\Phi_n\}$ .  $\square$

Thus, in what follows, we assume that the sequence  $\{\Phi_n\}$  is bounded.

**Lemma 2.** *Let  $\Psi_*$  be a bounded sequence of upper-triangular matrices:*

$$\Psi_n = \begin{pmatrix} \lambda_n & s_n \\ 0 & \frac{1}{\lambda_n} \end{pmatrix} \quad (4)$$

and

$$t_0 = \max\{|s_n/\lambda_n|, n \in \mathbb{N}\}. \quad (5)$$

*Then, for all  $t > t_0$  and all  $K > 0$ , there exists  $N \in \mathbb{N}$  such that, for each  $j \in \mathbb{N}$ , at least one of  $N$  products*

$$\Psi_{j+m} H(t) \cdot \dots \cdot \Psi_{j+1} H(t), \quad m = 1, 2, \dots, N \quad (6)$$

*has norm larger than  $K$ .*

*Proof.* Let us fix  $t > t_0$ . It follows by induction in  $m$  that

$$\Psi_{j+m}H(t) \cdot \dots \cdot \Psi_{j+1}H(t) = \begin{pmatrix} \Pi_{j,m} & t\Pi_{j,m}S_{j,m}(t) \\ 0 & \Pi_{j,m}^{-1} \end{pmatrix}$$

where

$$\Pi_{j,m} = \lambda_{j+1} \cdot \dots \cdot \lambda_{j+m}, \quad (7)$$

$$S_{j,m}(t) = \left( t + \frac{s_{j+1}}{\lambda_{j+1}} \right) + \frac{t + \frac{s_{j+2}}{\lambda_{j+2}}}{\lambda_{j+1}^2} + \dots + \frac{t + \frac{s_{j+m}}{\lambda_{j+m}}}{\lambda_{j+1}^2 \cdot \dots \cdot \lambda_{j+m-1}^2}. \quad (8)$$

Suppose that the assertion of the lemma is wrong. Then there exists  $K > 0$  such that for any  $N$  one can find  $j$  with the property that all products (6) have norm less than  $K$ . It follows that  $K^{-1} < |\Pi_{j,m}| < K$  and  $|S_{j,m}(t)| < \frac{K^2}{t}$ .

The denominators of the summands in the right hand side of (8) are equal to  $\Pi_{j,k}^2$ , so they do not exceed  $K^2$ . On the other hand,  $t + \frac{s_{j+1}}{\lambda_{j+1}} > t - t_0$ .

Hence

$$|S_{j,m}(t)| > m \frac{t - t_0}{K^2}.$$

Thus  $m \frac{t - t_0}{K^2} < \frac{K^2}{t}$ . In particular, this is true for  $m = N$ . But the inequality  $N \frac{t - t_0}{K^2} < \frac{K^2}{t}$  can not hold for all  $N$ . We obtained a contradiction.  $\square$

Let us say that a sequence  $\{a_n\}$  of complex numbers satisfies the condition (\*) if

$$\forall \varepsilon > 0 \quad \text{and} \quad \forall N \in \mathbb{N} \quad \exists i \in \mathbb{N} \quad \text{such that} \quad \max_{1 \leq j \leq N} |a_{i+j}| < \varepsilon. \quad (*)$$

**Lemma 3.** *Let  $\Psi_*$  be a bounded sequence of upper-triangular matrices. If a sequence of matrices  $\Phi_*$  is so close to  $\Psi_*$  that  $\{\|\Phi_n - \Psi_n\|\}$  satisfies (\*), then  $\mathfrak{B}(\Phi_*) \subseteq [0; t_0]$  with  $t_0$  the same as in (5).*

*Proof.* Assume, to the contrary, that for some  $t > t_0$  there exists  $M > 0$  such that  $\|P_n(t)\| \leq M$  for all  $n \in \mathbb{N}$ . Applying Lemma 2 to the sequence  $\Psi_*$  with  $K = 2M^2$ , we obtain a positive integer  $N$  such that, for any  $j$ , at least one of the products (6) with  $m \leq N$  has norm larger than  $2M^2$ .

Fix arbitrary  $\delta > 0$  and  $C > 1 + \sup \|\Phi_n\|$  and choose  $\varepsilon > 0$  with

$$\varepsilon < \frac{\delta(C-1)}{(C^N-1)\|H(t)\|^N}.$$

For these  $N$  and  $\varepsilon$ , find  $i$  according to condition (\*):

$$\|\Phi_{i+j} - \Psi_{i+j}\| < \varepsilon \quad j = 1, 2, \dots, N.$$

By our choice of  $N$ , there exists  $m$ ,  $1 \leq m \leq N$ , for which

$$\|\Psi_{i+m}H(t) \cdot \dots \cdot \Psi_{i+1}H(t)\| > K.$$

Now, we estimate the product  $\Phi_{i+m}H(t) \cdot \dots \cdot \Phi_{i+1}H(t)$ . On the one hand, it is close to  $\Psi_{i+m}H(t) \cdot \dots \cdot \Psi_{i+1}H(t)$ :

$$\begin{aligned} & \|\Phi_{i+m}H(t) \cdot \dots \cdot \Phi_{i+1}H(t) - \Psi_{i+m}H(t) \cdot \dots \cdot \Psi_{i+1}H(t)\| \\ & \leq \|\Phi_{i+m}H(t) \cdot \dots \cdot \Phi_{i+2}H(t)(\Phi_{i+1} - \Psi_{i+1})H(t)\| \\ & + \|\Phi_{i+m}H(t) \cdot \dots \cdot \Phi_{i+2}H(t)\Psi_{i+1}H(t) - \Psi_{i+m}H(t) \cdot \dots \cdot \Psi_{i+1}H(t)\| \\ & \leq \dots \leq \|\Phi_{i+m}H(t) \cdot \dots \cdot \Phi_{i+2}H(t)(\Phi_{i+1} - \Psi_{i+1})H(t)\| \\ & + \|\Phi_{i+m}H(t) \cdot \dots \cdot (\Phi_{i+2} - \Psi_{i+2})H(t)\Psi_{i+1}H(t)\| \\ & + \dots + \|(\Phi_{i+m} - \Psi_{i+m})H(t)\Psi_{i+m-1}H(t) \cdot \dots \cdot \Psi_{i+1}H(t)\| \\ & \leq \varepsilon(C^{m-1} + C^{m-2} + \dots + 1)\|H(t)\|^m \leq \frac{\varepsilon(C^m - 1)\|H(t)\|^m}{C - 1} < \delta. \end{aligned}$$

Therefore

$$\|\Phi_{i+m}H(t) \cdot \dots \cdot \Phi_{i+1}H(t)\| \geq \|\Psi_{i+m}H(t) \cdot \dots \cdot \Psi_{i+1}H(t)\| - \delta > 2M^2 - \delta. \quad (9)$$

On the other hand,

$$\|\Phi_{i+m}H(t) \cdot \dots \cdot \Phi_{i+1}H(t)\| = \|P_{i+m}(t) (P_i(t))^{-1}\|$$

$$\leq \|P_{i+m}(t)\| \cdot \|(P_i(t))^{-1}\| = \|P_{i+m}(t)\| \cdot \|P_i(t)\| \leq M^2.$$

which contradicts (9). □

We will also need two auxiliary results from the classical potential theory in the spirit of Wiener's criterion [2]. Let  $\Omega$  be a bounded domain in the complex plane,  $z_0 \in \Omega$ . Recall that the harmonic measure  $\omega$  on the boundary  $\partial\Omega$  respect to a point  $z_0$  is defined by the condition

$$u(z_0) = \int_{\partial\Omega} u(z) d\omega, \quad (10)$$

for all harmonic continuous function on  $\overline{\Omega}$ , and that for subharmonic  $u$  the equality should be changed by the inequality

$$u(z_0) \leq \int_{\partial\Omega} u(z) d\omega. \quad (11)$$

We will denote by  $|E|$  the Lebesgue measure of a set  $E \subset \mathbb{R}$ .

**Lemma 4.** *Let  $E$  be a closed subset of  $[1, +\infty]$  of infinite length with the property that*

$$|E \cap [a, 4a]| \leq 1 \quad \text{for each } a > 0. \quad (12)$$

*Fix  $z_0 \in \mathbb{C} \setminus E$ . Take  $R > |z_0|$  and consider the domain*

$$\Omega_R = \{z \in \mathbb{C} : |z| < 2R, z \notin E_{1,R}\},$$

*where by  $E_{a,b}$  we denote  $E \cap [a, b]$ . Then the harmonic measure  $\omega_R$  on  $\partial\Omega_R$ , associated with the point  $z_0$ , satisfies the condition*

$$\lim_{R \rightarrow \infty} \omega_R(T_R) \log(1 + 2R) = 0$$

*where  $T_R = \partial\Omega_R \cap \{|z| = 2R\}$ .*

*Proof.* Observe, first of all, that (12) implies that for each  $a > 0$ ,

$$\int_{E_{a,\infty}} \frac{dt}{t} \leq \sum_{k=0}^{\infty} \frac{1}{4^k a} |E_{4^k a, 4^{k+1} a}| \leq \frac{4}{3a}.$$

Consider the auxiliary potential

$$U(z) = \int_{E_{1,R}} \log \left| 1 - \frac{z}{t} \right| dt.$$



Notice that  $U \in C(\overline{\Omega_R})$ <sup>2</sup> and  $U$  is harmonic in  $\Omega_R$ . Since  $\partial\Omega_R$  consists of the circumference  $T_R$  of radius  $2R$  centered at 0 and the set  $E_{1,R}$ , we have

$$U(z_0) = \int_{T_R} U(z) d\omega_R(z) + \int_{E_{1,R}} U(z) d\omega_R(z). \quad (13)$$

Hence

$$\int_{T_R} U(z) d\omega_R(z) \leq |U(z_0)| - \int_{E_{1,R}} U(z) d\omega_R(z). \quad (14)$$

It follows from the definition of  $U$  that

$$|U(z_0)| \leq \int_{E_{1,R}} \log \left\{ 1 + \frac{|z_0|}{t} \right\} dt \leq |z_0| \int_{E_{1,R}} \frac{dt}{t} \leq \frac{4}{3} |z_0|. \quad (15)$$

Choose  $b > 0$  so that  $|E_{1,b}| = 1$ . We may suppose that  $R > 2 + b + |z_0|$ .

We claim that for every  $z \in E_{1,R}$ ,

$$\begin{aligned} U(z) &\geq \int_{E_{1,b}} \log \left| 1 - \frac{z}{t} \right| dt + \int_{E_{\frac{1}{2}z, 2z}} \log \left| 1 - \frac{z}{t} \right| dt + \\ &\quad \int_{E_{2z, +\infty}} \log \left| 1 - \frac{z}{t} \right| dt = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 \end{aligned}$$

The reason for this inequality is that  $\mathcal{J}_2$  and  $\mathcal{J}_3$  together give exactly the integral of *all* negative values of the function  $t \rightarrow \log \left| 1 - \frac{z}{t} \right|$  on  $E_{1,+\infty}$ , so the extension of the upper limit from  $R$  to  $+\infty$  in  $\mathcal{J}_3$  and possible overlapping with  $\mathcal{J}_1$  are not problems: essentially what is said is that the integral of a real-valued function over a set  $F$  is not less than its integral over any subset of  $F$  plus the integral of all its negative values over any superset of  $F$ .

Observe that, since  $\log |1 - x| \geq -2x$  for  $0 < x < 1/2$ , we have

$$\mathcal{J}_3 \geq -2z \int_{E_{2z, +\infty}} \frac{dt}{t} \geq -\frac{4}{3}$$

regardless of  $z$ . We also have (recall that  $|E_{1,b}| = 1$ )

$$\mathcal{J}_1 - \log z = \int_{E_{1,b}} \log \left| \frac{1}{z} - \frac{1}{t} \right| dt \rightarrow - \int_{E_{1,b}} \log t dt \quad \text{as } z \rightarrow +\infty.$$

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<sup>2</sup>This follows, for example, from the Continuity Principle (see [3, section 3.1])

Since  $\mathcal{J}_1 - \log z$  is a continuous function on  $[1, +\infty)$ , we get  $\mathcal{J}_1 \geq \log z - C_1$  where  $C_1$  is some large constant independent on  $R$ . Next,

$$\begin{aligned} \mathcal{J}_2 &= \int_{E_{\frac{1}{2}z, 2z}} \log |t - z| \, dt - \int_{E_{\frac{1}{2}z, 2z}} \log t \, dt \geq \\ &\quad \int_{[z-1, z+1]} \log |t - z| \, dt - \log(2z) |E_{\frac{1}{2}z, 2z}| \geq -2 - \log z - \log 2 \end{aligned}$$

(here, we used the inequality  $|E_{\frac{1}{2}z, 2z}| \leq 1$ ). Therefore,  $U(z) \geq -C_1 - 10/3 - \log 2 = -C_2$  on  $E_{1,R}$  and, taking into account that  $\omega_R(E_{1,R}) \leq \omega_R(\partial\Omega) = 1$ , we get

$$\int_{E_{1,R}} U(z) \, d\omega_R(z) \geq -C_2 \omega_R(E_{1,R}) \geq -C_2. \quad (16)$$

Now by (14),

$$\int_{T_R} U(z) d\omega_R(z) \leq C_3, \quad (17)$$

where  $C_3 = C_2 + \frac{4}{3}|z_0|$ .

The last observation we need is that for  $z \in T_R$ ,

$$U(z) \geq \int_{E_{1,R}} \log \left| 1 - \frac{2R}{t} \right| dt. \quad (18)$$

So (17) gives

$$\omega_R(T_R) \leq C_3 \left\{ \int_{E_{1,R}} \log \left| 1 - \frac{2R}{t} \right| dt \right\}^{-1}.$$

Thereby,

$$\begin{aligned} \omega_R(T_R) \log(1 + 2R) &\leq C_3 \left\{ \frac{1}{\log(1 + 2R)} \int_{E_{1,R}} \left| 1 - \frac{2R}{t} \right| dt \right\}^{-1} \\ &= \left\{ \int_{E_{1,R}} L_R(t) dt \right\}^{-1}. \end{aligned}$$

Note that  $0 \leq L_R(t) \leq 1$  for each  $t \in E_{1,R}$  and  $L_R(t) \rightarrow 1$  as  $R \rightarrow +\infty$  for every fixed  $t \in E$ . Thereby,  $\int_{E_{1,R}} L_R(t) dt \rightarrow |E| = +\infty$ , and we are done.  $\square$

**Lemma 5.** *Let  $u_n(z)$  be a sequence of continuous subharmonic functions satisfying the estimate  $u_n(z) \leq \log(1 + |z|) + A$  for all  $n \geq 1$ ,  $z \in \mathbb{C}$  and some  $A > 0$ . If  $E \subset \mathbb{R}$  has infinite length and  $\limsup_{n \rightarrow \infty} u_n(z) \leq 0$  for all  $z \in E$ , then  $\limsup_{n \rightarrow \infty} u_n(z) \leq 0$  for all  $z \in \mathbb{C}$ .*

*Proof.* Since every measurable set of infinite length contains a closed subset of infinite length we may assume without loss of generality that  $E$  is closed. Also we may assume that  $|E \cap [1, +\infty)| = +\infty$ . Indeed, otherwise  $|E \cap [-\infty, -1]| = +\infty$  and we may consider the set  $-E$  and functions  $u_n(-z)$  instead. Thus we can always assume that  $E \subset [1, +\infty]$ . The last regularization we need is the following. Take any dyadic interval  $I_k = [2^{k-1}, 2^k]$  ( $k = 1, 2, \dots$ ). If  $|E \cap I_k| < 1/3$ , leave the corresponding piece of  $E$  alone. Otherwise replace it by some subset of length exactly  $1/3$ . The resulting set still has infinite length. Indeed if we made finitely many replacements, we dropped only a set of finite length from  $E$ , and if we made infinitely many replacements, we have infinitely many disjoint pieces of length  $1/3$  in the resulting set. After such regularization, the set  $E$  enjoys the property (12). We will use the notation introduced in the previous Lemma.

Choose  $z_0 \in \mathbb{C}$ ; we have to prove that  $\limsup_{n \rightarrow \infty} u_n(z_0) \leq 0$ . This is evident if  $z_0 \in E$ , so we assume that  $z_0 \in \mathbb{C} \setminus E$ .

For  $R > |z_0|$ , we have, by (11),

$$u_n(z_0) \leq \int_{\partial\Omega_R} u_n(z) d\omega_R(z) = \int_{T_R} u_n(z) d\omega_R(z) + \int_{E_{1,R}} u_n(z) d\omega_R(z).$$

Note that, for fixed  $R$ , the length of  $E_{1,R}$  is finite,  $u_n$  are uniformly bounded from above on  $E_{1,R}$ , and  $\limsup_{n \rightarrow \infty} u_n(z) \leq 0$  for all  $z \in E_{1,R}$ . Therefore, the Fatou lemma yields

$$\limsup_{n \rightarrow \infty} \int_{E_{1,R}} u_n(z) d\omega_R(z) \leq 0$$

and thereby

$$\limsup_{n \rightarrow \infty} u_n(z_0) \leq \sup_{n \geq 1} \int_{T_R} u_n(z) d\omega_R(z) \leq \omega_R(T_R)(\log(1 + 2R) + A)$$

for any fixed  $R$ . By Lemma 4, the result follows.  $\square$

## 4 The proof of Theorem 1

Now we can prove Theorem 1.

*Proof.* We use Iwasawa's decomposition of  $2 \times 2$  matrices:

$$\begin{aligned}\Phi_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} &= \begin{pmatrix} 1 & s_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_n & 0 \\ 0 & \frac{1}{\lambda_n} \end{pmatrix} \begin{pmatrix} \cos \alpha_n & -\sin \alpha_n \\ \sin \alpha_n & \cos \alpha_n \end{pmatrix} \\ &= H(s_n)D(\lambda_n)R(\alpha_n),\end{aligned}$$

where  $\alpha_n = \arcsin \frac{c_n \operatorname{sign}(d_n)}{\sqrt{c_n^2 + d_n^2}} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ,  $s_n = \frac{a_n c_n + b_n d_n}{c_n^2 + d_n^2}$ ,  $\lambda_n = \frac{\operatorname{sign}(d_n)}{\sqrt{c_n^2 + d_n^2}}$ .

By Lemma 1, we may assume that the system  $\Phi_*$  is bounded. It follows that both sequences  $\{H(s_n)\}$  and  $\{D(\lambda_n)\}$  are bounded. Indeed,

$$s_n = \frac{(a_n, b_n) \cdot (c_n, d_n)}{\|(c_n, d_n)\|^2} \leq \frac{\|(a_n, b_n)\|}{\|(c_n, d_n)\|} = \frac{\|(a_n, b_n)\|}{\|(d_n, -c_n)\|} \leq \frac{\|(a_n, b_n)\|}{1/\|(a_n, b_n)\|} \leq C^2$$

where  $C > 1 + \sup\{\|\Phi_n\|\}$ . Thus  $\{H(s_n)\}$  is a bounded sequence. The boundedness of  $\{D(\lambda_n)\}$  follows from the equality

$$D(\lambda_n) = H(s_n)^{-1} \Phi_n R(\alpha_n)^{-1}.$$

Now denote

$$q = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n |\alpha_j|$$

and consider two cases separately:  $q = 0$  and  $q > 0$ . In both cases we will prove stronger statements than the assertion of Theorem 1.

**Case A:**  $q = 0$ . We will show that in this case the set  $\mathfrak{B}(\Phi_*)$  is bounded. Note, that in this case the sequences  $\{\alpha_n\}$  and hence  $\{\sin \alpha_n\}$  satisfy condition (\*).

Since

$$\begin{aligned}\|\Phi_n - H(s_n)D(\lambda_n)\| &\leq \|H(s_n)\| \|D(\lambda_n)\| \left\| \begin{pmatrix} \cos(\alpha_n) - 1 & -\sin \alpha_n \\ \sin \alpha_n & \cos(\alpha_n) - 1 \end{pmatrix} \right\| \\ &\leq 2|\sin \alpha_n| \|H(s_n)\| \|D(\lambda_n)\|,\end{aligned}$$

the sequence  $\Phi_*$  is close in (\*)-sense to the sequence of upper-triangular matrices  $\{H(s_n)D(\lambda_n)\}$ , i.e, the sequence of norms  $\|\Phi_n - H(s_n)D(\lambda_n)\|$  satisfies

(\*). Thus, according to Lemma 3, the sequence of evolutions  $\{P_n(t)\}$  is unbounded for every  $t > t_0$ , where

$$t_0 = \max\{|s_n/\lambda_n^2|, n \in \mathbb{N}\}.$$

This completes the proof in the case A.  $\square$

**Case B:**  $q > 0$ . Extending our problem to the complex plane, we consider  $SL(2, \mathbb{C})$ -matrices  $H(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$  instead of  $H(t)$ . Respectively,  $P_n(z) = \Phi_n H(z) \dots \Phi_1 H(z)$ . We will show that the set

$$E = \{z \in \mathbb{C} : \limsup_{n \rightarrow \infty} \frac{\log \|P_n(z)\|}{n} = 0\}$$

is contained in  $\mathbb{R}$  and has finite length. So we not only prove that the sequence  $P_*$  is unbounded but prove that it has exponential growth for all  $t$  apart of a set of finite measure.

Our first task is to show that

$$\limsup_{n \rightarrow \infty} \frac{\log \|P_n(z)\|}{n} > 0, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (19)$$

Assume that  $\text{Im}z > 0$  (the case  $\text{Im}z < 0$  can be considered in a similar way). Let us consider the quadratic form

$$Q(x) = \text{Im}(x_1 \bar{x}_2), \quad x = (x_1, x_2) \in \mathbb{C}^2.$$

For any matrix  $A \in SL(2, \mathbb{R})$  and  $x \in \mathbb{C}^2$ , one has  $Q(Ax) = Q(x)$ . On the other hand, for every  $z \in \mathbb{C}$  with  $\text{Im}z > 0$ , one has

$$Q(H(z)x) = Q(x) + \text{Im}z |x_2|^2 \geq Q(x) \left(1 + \text{Im}z \frac{|x_2|}{|x_1|}\right).$$

Now, we need one more lemma.

**Lemma 6.** *Let  $\alpha_n \in [-\pi/2, \pi/2]$ , then there exists  $k \geq 1$  such that for all  $n \in \mathbb{N}$*

$$Q(H(z)H(s_n)D(\lambda_n)R(\alpha_n)H(z)x) \geq Q(x) \left(1 + \frac{|\alpha_n| \text{Im}z}{2k(1 + |z|)}\right). \quad (20)$$

*Proof of Lemma 6.* We split the proof into two cases.

**Case 1:**  $\frac{|x_2|}{|x_1|} \geq \frac{|\alpha_n|}{2(1+|z|)}$ . Then

$$\begin{aligned} Q(H(z)H(s_n)D(\lambda_n)R(\alpha_n)H(z)x) \\ \geq Q(H(s_n)D(\lambda_n)R(\alpha_n)H(z)x) &= Q(H(z)x) \\ \geq Q(x) \left(1 + \operatorname{Im} z \cdot \frac{|x_2|}{|x_1|}\right) &\geq Q(x) \left(1 + \frac{|\alpha_n| \operatorname{Im} z}{2(1+|z|)}\right). \end{aligned}$$

**Case 2:** Now, we suppose that  $\frac{|x_2|}{|x_1|} \leq \frac{|\alpha_n|}{2(1+|z|)}$ , and split the proof of the estimate (20) into 4 steps.

1. Let us estimate how the matrix  $H(z)$  changes the ratio of coordinates. Since  $|\alpha_n| \leq \frac{\pi}{2}$  we have

$$\begin{aligned} |[H(z)x]_2| &= |x_2| \leq \frac{|\alpha_n|}{2} |x_1| \frac{1}{1+|z|} \\ &= \frac{|\alpha_n|}{2} \cdot |x_1| \cdot \left(1 - \frac{|z|}{1+|z|}\right) \leq \frac{|\alpha_n|}{2} \cdot |x_1| \cdot \left(1 - \frac{|\alpha_n||z|}{2(1+|z|)}\right) \end{aligned}$$

where  $[\ ]_2$  means the second coordinate. Next,

$$|[H(z)x]_1| = |x_1 + zx_2| \geq |x_1| \left|1 - |z| \frac{|x_2|}{|x_1|}\right| > |x_1| \cdot \left(1 - \frac{|\alpha_n||z|}{2(1+|z|)}\right)$$

since  $|z| \frac{|x_2|}{|x_1|} \leq \frac{|\alpha_n|}{2} \frac{|z|}{1+|z|} < \frac{|\alpha_n|}{2} < 1$ . Thus

$$|[H(z)x]_2| \leq \frac{|\alpha_n|}{2} |[H(z)x]_1|.$$

2. It is easy to check the following property of an orthogonal matrix  $R(\alpha)$ : for any  $x \in \mathbb{C}^2$  and  $|\alpha| \leq \frac{\pi}{2}$ , the inequality  $|x_2| \leq \frac{|\alpha|}{2}|x_1|$  implies  $|[R(\alpha)x]_2| \geq \frac{|\alpha|}{2}|[R(\alpha)x]_1|$ . Therefore

$$|[R(\alpha_n)H(z)x]_2| \geq \frac{|\alpha_n|}{2} |[R(\alpha_n)H(z)x]_1|.$$

Denote temporarily  $R(\alpha_n)H(z)x = y$ , then  $|y_2| \geq \frac{|\alpha_n|}{2} \cdot |y_1|$ .

3. We set  $k = C^2$  with  $C = \sup_i \|\Phi_i\|$  and obtain

$$\frac{|[D(\lambda_n)y]_2|}{|[D(\lambda_n)y]_1|} = \frac{\left|\frac{1}{\lambda_n}\right| |y_2|}{|\lambda_n| |y_1|} \geq \frac{1}{\lambda_n^2} \cdot \frac{|\alpha_n|}{2} > \frac{|\alpha_n|}{2k}.$$

4. At last

$$\begin{aligned} Q(H(z)H(s_n)D(\lambda_n)R(\alpha_n)H(z)x) &= Q(H(z)H(s_n)D(\lambda_n)y) \\ &= Q(H(s_n)D(\lambda_n)y) + \operatorname{Im} z |[H(s_n)D(\lambda_n)y]_2|^2 = Q(D(\lambda_n)y) \\ &\quad + \operatorname{Im} z |[D(\lambda_n)y]_2|^2 \geq Q(D(\lambda_n)y) \left(1 + \operatorname{Im} z \cdot \frac{|[D(\lambda_n)y]_2|}{|[D(\lambda_n)y]_1|}\right) \\ &\geq Q(D(\lambda_n)y) \left(1 + \frac{|\alpha_n| \operatorname{Im} z}{2k}\right) \geq Q(x) \left(1 + \frac{|\alpha_n| \operatorname{Im} z}{2k}\right) \\ &> Q(x) \left(1 + \frac{|\alpha_n| \operatorname{Im} z}{2k(1 + |z|)}\right). \end{aligned}$$

□

Returning to the proof of the theorem, we have

$$P_n(z) = H(s_n)D(\lambda_n)R(\alpha_n)H(z) \cdot \dots \cdot H(s_1)D(\lambda_1)R(\alpha_1)H(z) = \widehat{\prod_{1 \leq j \leq n} H(s_j)D(\lambda_j)R(\alpha_j)H(z)}$$

(recall that  $\widehat{\prod_{1 \leq j \leq n} A_j}$  stands for the matrix product  $A_n A_{n-1} \dots$ ).

Denote

$$B_n(z) = H(z/2)P_n(z)H(z/2)^{-1} = \widehat{\prod_{1 \leq j \leq n} H(z/2)H(s_j)D(\lambda_j)R(\alpha_j)H(z/2)}.$$

Then

$$\overline{\lim_{n \rightarrow \infty}} \frac{\log \|P_n(z)\|}{n} = \overline{\lim_{n \rightarrow \infty}} \frac{\log \|B_n(z)\|}{n}.$$

Let us consider the vector  $x = \left( \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$ ,  $|x|^2 = 2Q(x) = 1$ . Then, due to the fact that for arbitrary  $y$

$$\|y\|^2 = y_1^2 + y_2^2 \geq 2|y_1 y_2| = 2|y_1 \overline{y_2}| \geq 2\operatorname{Im}(y_1 \overline{y_2}) = 2Q(y),$$

we obtain

$$\begin{aligned} \log \|B_n(z)\| &\geq \frac{1}{2} \log \|B_n(z)x\|^2 \geq \frac{1}{2} \log (2Q(B_n(z)x)) \\ &\geq \frac{1}{2} \sum_{j=1}^n \log \left( 1 + \frac{|\alpha_j| \operatorname{Im} \frac{z}{2}}{2k(1 + |\frac{z}{2}|)} \right) \geq \frac{1}{4} \sum_{j=1}^n \frac{|\alpha_j| \operatorname{Im} \frac{z}{2}}{2k(1 + |\frac{z}{2}|)} = \frac{\operatorname{Im} \frac{z}{2}}{8k(1 + |\frac{z}{2}|)} \sum_{j=1}^n |\alpha_j| \end{aligned}$$

where we used that  $\log(1+x) \geq \frac{1}{2}x$  for  $x \in (0; 1)$  and  $\frac{|\alpha_j| \operatorname{Im} z}{2k(1 + |z|)} \leq \frac{|\alpha_j|}{2k} < 1$ .

As a consequence, we obtain:

$$\overline{\lim}_{n \rightarrow \infty} \frac{\log \|P_n(z)\|}{n} \geq \overline{\lim}_{n \rightarrow \infty} \frac{\operatorname{Im} \frac{z}{2}}{8k(1 + |\frac{z}{2}|)} \cdot \frac{1}{n} \sum_{j=1}^n |\alpha_j| = \frac{q \cdot \operatorname{Im} \frac{z}{2}}{8k(1 + |\frac{z}{2}|)} > 0.$$

This proves (19), that is the exponential growth of  $\|P_n(z)\|$  for non-real  $z$ . Thus  $E \subset \mathbb{R}$ .

The claim that  $|E| < \infty$  follows now from Lemma 5 applied to the subharmonic functions

$$u_n(z) = \frac{\log \|P_n(z)\|}{n}.$$

Indeed, the norm of the matrix  $H(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$  does not exceed  $1 + |z|$ . Hence

$$\|P_n(z)\| = \left\| \prod_{1 \leq j \leq n} [\Phi_j H(z)] \right\| \leq \|H(z)\|^n \cdot \prod_{1 \leq j \leq n} \|\Phi_j\| \leq (1 + |z|)^n \cdot k^n$$

Therefore

$$\frac{1}{n} \log \|P_n(z)\| \leq \log(1 + |z|) + \log k.$$

Thus the functions  $u_n$  satisfy the majorization condition of Lemma 5. By definition,  $\limsup u_n(z) = 0$  for  $z \in E$ . If  $|E| = \infty$  then Lemma 5 implies that  $\limsup u_n(z) \leq 0$  for all  $z \in \mathbb{C}$ , however, this contradicts (19).  $\square$



## 5 Constructing an exceptional set containing a given sequence

Let, as above,  $H(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ ,  $\Phi_*$  be a sequence of matrices from  $SL(2, \mathbb{R})$ . The exceptional set was defined as follows

$$\mathfrak{B}(\Phi_*) = \{t \geq 0 : \sup_K \left\| \prod_{1 \leq k \leq K}^{\circlearrowleft} \Phi_k H(t) \right\| < \infty\}$$

We have proved that this set always has finite measure. Nevertheless it can be unbounded. Moreover, it can contain an arbitrary given sequence:

**Theorem 2.** *For every sequence  $\{t_n\}$  of positive numbers there exists a sequence  $\Phi_*$  such that  $\{t_n\} \subset \mathfrak{B}(\Phi_*)$ .*

*Proof.* First let us note the following fact: for every  $SL(2, \mathbb{R})$ -matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  there exists an orthogonal matrix  $R = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$  for which  $(RA)^2 = -\mathbf{1}$ . To prove it is sufficient to choose  $R$  such that  $\text{tr}(RA) = 0$ , for example to take  $\alpha = \text{arctg} \frac{c-b}{a+d}$ .

Now let us construct two sequences  $\{A_n(t)\}$  and  $\{R_n\}$  in the following way. For  $A_1(t) = H(t)$  we will choose  $R_1$  such that  $(R_1 A_1(t_1))^2 = -\mathbf{1}$ . Further for each  $n \in \mathbb{N}$  we define  $A_n(t) = (R_{n-1} A_{n-1}(t))^2$  and we choose  $R_n$  such that  $(R_n A_n(t_n))^2 = -\mathbf{1}$ .

Now we define sequence  $\Phi_*$  as follows:  $\Phi_n = R_{k+1} \dots R_2 R_1$  where  $k$  is the the largest  $j$  such that  $2^j$  divides  $n$ . Thus we have:  $\Phi_1 = R_1$ ,  $\Phi_2 = R_2 R_1$ ,  $\Phi_3 = R_1$ ,  $\Phi_4 = R_3 R_2 R_1$ ,  $\Phi_5 = R_1$ ,  $\Phi_6 = R_2 R_1$ ,  $\Phi_7 = R_1$ ,  $\Phi_8 = R_4 R_3 R_2 R_1$ ,  $\Phi_9 = R_1$ , ...

Then the evolution sequence  $P_n(t) = \Phi_n H(t) \Phi_{n-1} H(t) \dots \Phi_1 H(t)$  has the form

$$\begin{aligned} & \dots R_3 R_2 R_1 H(t) R_1 H(t) R_2 R_1 H(t) R_1 H(t) R_3 R_2 R_1 H(t) R_1 H(t) R_2 R_1 H(t) R_1 H(t) \\ &= \dots R_2 A_2(t) R_4 R_3 R_2 A_2(t) R_2 A_2(t) R_3 R_2 A_2(t) R_2 A_2(t) \\ &= \dots R_4 R_3 A_3(t) R_3 A_3(t) R_4 R_3 A_3(t) R_3 A_3(t) \end{aligned}$$

and so on. Thus, for any  $k$ ,

$$P_n(t) = B(n, k) \dots R_{k+2} R_{k+1} R_k A_k(t) R_k A_k(t) R_{k+1} R_k A_k(t) R_k A_k(t)$$

where the factor  $B(n, k)$  is a product of not more than  $N = N(k)$  matrices which are either orthogonal or equal to  $H(t)$ . It follows that the norm of  $B(n, k)$  does not exceed a constant depending only on  $k$  and  $t$ :

$$\|B(n, k)\| \leq C(k, t).$$

Since  $(R_k A_k(t))^2 = -\mathbf{1}$  for  $t = t_k$ , one has

$$\|P_n(t_k)\| \leq C(k, t_k).$$

This means that  $t_k \in \mathfrak{B}(\Phi_*)$ . □

## 6 Constructing an essentially unbounded exceptional set

In this section we will construct an example of a sequence  $\Phi_* \subset SL(2, \mathbb{R})$  for which the exceptional set

$$\mathfrak{B}(\Phi_*) = \{t \geq 0 : \sup_K \left\| \prod_{1 \leq k \leq K}^{\curvearrowright} \Phi_k H(t) \right\| < \infty\}$$

is essentially unbounded, that is  $|\mathfrak{B}(\Phi_*) \cap [a, +\infty)| > 0$  for all  $a > 0$ .

Let us consider a sequence of matrices  $M_j = M(c_j) = \begin{pmatrix} 1 & 0 \\ c_j & 1 \end{pmatrix}$  ( $j \geq 0$ ) with  $c_j \neq 0$ . We define the sequence  $\{\Phi_k\}$  ( $k \geq 1$ ) in the following way:  $\Phi_k = M(c_{j(k)})$  where  $j(k)$  is the largest  $j$  such that  $2^j$  divides  $k$ . The first few terms of the sequence  $\Phi_*$  are

$$M_0, M_1, M_0, M_2, M_0, M_1, M_0, M_3, M_0, M_1, M_0, M_2, M_0, M_1, M_0, \dots$$

(“the abacaba order” [4]).

**Theorem 3.** *There exists a sequence  $\{c_j\}$  such that the set  $\mathfrak{B}(\Phi_*)$  is essentially unbounded.*

The proof of this statement will be given in the next section. Here we only outline its basic ideas.

First of all note that our choice of the sequence  $\Phi_k$  implies that the partial products  $\prod_{1 \leq k \leq K} \Phi_k H(t)$  with diadic numbers ( $K = 2^m$ ) are related by a simple recurrent formula. Namely let us define a sequence of matrix-functions  $A_n(t)$ ,  $n \geq -1$ , as follows:

$$A_{-1}(t) = H(t), \quad A_{n+1}(t) = A_n(t)M(c_{n+1})A_n(t).$$

Then it is easy to check that  $\prod_{1 \leq k \leq 2^m} \Phi_k H(t) = M(c_m)A_{m-1}(t)$ . More gener-

ally,  $\prod_{2^{l+1} \leq k \leq 2^{l+m}} \Phi_k H(t) = M(c_m)A_{l-1}(t)$ . So it is possible to express all partial products via  $A_n(t)$ . For example, for  $K = 84$ , we have  $84 = 2^6 + 2^4 + 2^2$  and, respectively,

$$\prod_{1 \leq k \leq 84} \Phi_k H(t) = M(c_2)A_1(t)M(c_4)A_3(t)M(c_6)A_5(t).$$

The general formula is

$$\prod_{1 \leq k \leq K} \Phi_k H(t) = \prod_{1 \leq \ell \leq L} M_{j_\ell} A_{j_\ell-1}(t) \quad (21)$$

(here  $\{j_\ell\}$  is the strictly increasing finite sequence of integers such that  $K = 2^{j_1} + 2^{j_2} + \dots + 2^{j_L}$ ). It follows from (21) that for proving the boundedness of the sequence of all partial products for a given  $t$ , it will be sufficient to find

an upper bound for the norms of partial products  $\prod_{1 \leq k \leq K} \Phi_k H(t)$  for all  $K$  that are multiples of  $2^m$  for some integer  $m$ . Note that only  $A_n$  with  $n \geq m$  can appear in such partial products.

Suppose that for some  $t > 0$  and for some integer  $m$  the following condition holds:

$$-2 < \operatorname{tr}(A_n(t)) < 2, \quad \text{for all } n \geq m. \quad (22)$$

Then the eigenvalues of  $A_n(t)$  are complex conjugate and the matrices are similar to diagonal ones:

$$A_n(t) = S_n(t)D_n(t)S_n(t)^{-1} \text{ where } D_n(t) = \begin{pmatrix} \lambda_n(t) & 0 \\ 0 & \overline{\lambda_n(t)} \end{pmatrix}.$$

Suppose also that, for all  $n \geq m$ ,

$$\|S_{n+1}(t) - S_n(t)\| < \varepsilon_n \quad (23)$$

where the numbers  $\varepsilon_n$  are such that  $\sum_{n=m}^{\infty} \varepsilon_n < \infty$ .

Under these conditions it can be proved, using (21), that all products

$\prod_{1 \leq k \leq K}^{\circlearrowleft} \Phi_k H(t)$  with  $K \in 2^{m+1}\mathbb{Z}$  are bounded by the same constant.

Thus, it remains to construct an essentially unbounded set  $E$  such that the conditions (22) and (23) are satisfied for all  $t \in E$  (with  $m$  depending on  $t$ ). We will define  $E$  as  $\bigcup_{k=0}^{\infty} \bigcap_{n=k}^{\infty} E_n$  where sets  $E_n$  are constructed inductively.

The possibility of the induction steps is provided by two auxiliary results (Lemmas 7 and 8) which state that (under some conditions) a set  $F \subset \mathbb{R}$  on which the condition (22) holds, can be slightly reduced and, respectively, can be extended by adding an interval located arbitrarily far away from the origin in such a way that on the new set the inequalities of type (23) hold.

For the beginning of the induction process we take a closed interval  $E_0 \subset (0, +\infty)$  such that  $\text{tr } A_0(t) \in (-2, 2)$  on  $E_0$ , and choose a sequence  $\{\varepsilon_n\}$  with  $\sum_{n=1}^{\infty} \varepsilon_n < \frac{1}{3}|E_0|$ .

Then we choose (using Lemma 7) a closed subset  $\tilde{E}_0 \subset E_0$  such that  $|E_0 \setminus \tilde{E}_0| < \varepsilon_1|E_0|$  and the conditions  $\text{tr } A_1(t) \in (-2, 2)$  and  $\|S_1(t) - S_0(t)\| < \varepsilon_1$  hold on  $\tilde{E}_0$ .

Now using Lemma 8 we find a closed interval  $I_1$  such that its left endpoint is larger than  $\sup E_0$  and  $\text{tr } A_1(t) \in (-2, 2)$  on  $I_1$ . We put  $E_1 = \tilde{E}_0 \cup I_1$ .

In this manner we proceed with constructing the intervals  $I_n$  and sets  $E_n$ . Namely, on the  $n$ -th step we get a set  $E_n$  such that  $I_n \subset E_n$ , and choose its subset  $\tilde{E}_n$ , satisfying  $|E_n \setminus \tilde{E}_n| < \varepsilon_n|I_n|$ , in such a way that the conditions  $\text{tr } A_j(t) \in (-2, 2)$  and  $\|S_j(t) - S_{j-1}(t)\| < \varepsilon_n$  are fulfilled for  $t \in \tilde{E}_n$  and for all  $j \leq n$ . Then we set  $E_{n+1} = E_n \cup I_{n+1}$  where the left endpoint of  $I_{n+1}$  is larger than  $\sup E_n$ .

The smallness of the deleted parts of the sets  $E_n$  and the condition  $I_n \subset (n, \infty)$  provide the essential unboundedness of the set  $E$ .

## 7 Proof of Theorem 3

We start with two auxiliary results.

Let us call a real polynomial matrix function  $A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}$  and a compact set  $E \subset (0, +\infty)$  a good pair if the following conditions hold:

- (i)  $\det A(t) = 1$  for all  $t$ ;
- (ii)  $\operatorname{tr} A(t) = a_{11}(t) + a_{22}(t)$  is a non-constant polynomial;
- (iii)  $\operatorname{tr} A(t) \in (-2, 2)$  for all  $t \in E$ .

If  $(A(t), E)$  is a good pair, then, according to the spectral theorem, one can find continuous functions  $\lambda : E \rightarrow \mathbb{T}$ ,  $\operatorname{Im} \lambda \neq 0$  and  $S : E \rightarrow SL(2, \mathbb{C})$  such that  $A(t) = S(t)D(t)S(t)^{-1}$  where  $D(t) = \begin{pmatrix} \lambda(t) & 0 \\ 0 & \frac{0}{\lambda(t)} \end{pmatrix}$ .

Choosing  $c \in \mathbb{R}$ , we set  $\tilde{A}(t) := A(t)M(c)A(t)$ .

**Lemma 7.** *Assume that  $(A(t), E)$  is a good pair. Let  $\varepsilon > 0$ . Then there exists  $\delta > 0$  such that, for every real  $c$  with  $|c| < \delta$ , there exists a compact set  $\tilde{E} \subset E$  such that*

- (a)  $|E \setminus \tilde{E}| < \varepsilon$ ;
- (b) *the pair  $(\tilde{A}(t), \tilde{E})$  is good;*

and

- (c) *A matrix-function  $\tilde{S}(t)$ , diagonalizing  $\tilde{A}(t)$ :*

$$\tilde{A}(t) = \tilde{S}(t)\tilde{D}(t)\tilde{S}(t)^{-1}$$

can be chosen in such a way that  $\|\tilde{S}(t) - S(t)\| < \varepsilon$ .

*Proof.* Note that if  $|c|$  is small then the function  $\tilde{A}(t)$  is close to  $A(t)^2$  on  $E$ :

$$\|\tilde{A}(t) - A(t)^2\| = \|A(t)[M(c) - I]A(t)\| \leq |c| \cdot \|A(t)\|^2 \leq C_1\delta$$

It follows that it is “almost diagonalized” by means of  $S(t)$

$$\|S(t)^{-1}\tilde{A}(t)S(t) - D(t)^2\| \leq C_1\delta \cdot \|S(t)\|^2 \leq C_2\delta$$

on  $E$ . Since the matrix function  $B(t) = S(t)^{-1}\tilde{A}(t)S(t)$  is similar to  $\tilde{A}(t)$  (so has the same diagonal part  $\tilde{D}(t)$ ), the pair  $(\tilde{A}(t), \tilde{E})$  (whatever  $\tilde{E}$  be chosen) is good if and only if  $(B(t), \tilde{E})$  is good. So we will deal with  $B(t)$ . Let us first of all show that  $B(t)$  can be diagonalized

$$B(t) = V(t)\tilde{D}(t)V(t)^{-1}$$

via a matrix  $V(t)$  which is close to  $I$ . It will follow that  $\tilde{A}(t) = S(t)V(t)\tilde{D}(t)V(t)^{-1}S(t)^{-1}$ , and  $\tilde{S}(t) = S(t)V(t)$  is close to  $S(t)$ .

We already have that

$$\|B(t) - D(t)^2\| \leq C_2\delta$$

on  $E$ . Now,  $D(t)^2 = \begin{pmatrix} \lambda^2(t) & 0 \\ 0 & \overline{\lambda^2(t)} \end{pmatrix}$  is a continuous diagonal matrix-function with distinct diagonal elements for all  $t \in E$  except, maybe, finitely many  $t$  satisfying the equation  $\text{tr } A(t) = 0$  (in which case  $\lambda(t) = \pm i$  and  $\lambda^2(t) = \overline{\lambda^2(t)} = -1$ ). Let  $G$  be any open set containing those exceptional  $t$  and such that  $|G| < \varepsilon$ . Put  $\tilde{E} = E \setminus G$ . Then  $\text{Tr } [D(t)^2] = 2\text{Re } [\lambda^2(t)] \subset [-a, a]$  for all  $t \in \tilde{E}$  with some  $a < 2$ . Let  $\delta > 0$  be so small that  $2C_2\delta < 2 - a$ . Then  $\text{tr } B(t) \in (-2, 2)$  and, therefore, the eigenvalues of  $B(t)$  are  $\tilde{\lambda}(t)$  and  $\overline{\tilde{\lambda}(t)}$  with  $|\tilde{\lambda}(t)| = 1$ . Moreover,  $\tilde{\lambda}(t)$  is a continuous function of  $t$  and  $|\tilde{\lambda}(t) - \lambda^2(t)| \leq C_3\sqrt{\delta}$ . Let now  $m = \min_{t \in \tilde{E}} |\text{Im } [\lambda^2(t)]|$  and note that  $m > 0$ . Also, let

$$B(t) - D(t)^2 =: \Delta(t) = \begin{pmatrix} \Delta_{11}(t) & \Delta_{12}(t) \\ \Delta_{21}(t) & \Delta_{22}(t) \end{pmatrix}.$$

Then the matrix  $V(t)$  whose columns are eigenvectors of  $B(t)$  is

$$V(t) = \begin{pmatrix} \overline{\lambda^2(t)} - \tilde{\lambda}(t) + \Delta_{22}(t) & \Delta_{12}(t) \\ -\Delta_{21}(t) & \overline{\tilde{\lambda}(t)} - \lambda^2(t) - \Delta_{11}(t) \end{pmatrix}.$$

The exact formula for  $V(t)$  doesn't matter but it is important that  $V(t)$  can be chosen to be a continuous function of  $t$  that is close to a diagonal matrix with equal non-zero elements on the diagonal when the perturbation  $\Delta(t)$  is close to 0. Note that  $\det V(t) \neq 0$  if  $\delta$  is small enough. Let now  $\tilde{V}(t) = \frac{1}{\sqrt{\det V(t)}} V(t)$  where the branch of the square root of the determinant is chosen in such a way that it equals  $\overline{\lambda^2(t)} - \lambda^2(t)$  when  $\Delta(t) = 0$ . Then the norm  $\|\tilde{V}(t) - I\|$  can be made arbitrarily small if  $\delta$  is small enough. It remains to note that  $B(t) = \tilde{V}(t)\tilde{D}(t)\tilde{V}(t)^{-1}$  where  $\tilde{D}(t) = \begin{pmatrix} \tilde{\lambda}(t) & 0 \\ 0 & \overline{\tilde{\lambda}(t)} \end{pmatrix}$ , so we can put  $\tilde{S}(t) = S(t)\tilde{V}(t)$ .

We proved the statement (c) of the lemma. The statement (a) follows from the inequality  $|G| < \varepsilon$ . To have (b) we must check conditions (i – iii) for the matrix  $\tilde{A}(t)$ . The condition (i) is obvious because the product of three matrices of determinant 1 is still a matrix of determinant 1. The condition (iii) is proved above: we have shown that  $\text{tr } B(t) \subset (-2, 2)$  but  $\text{tr } \tilde{A}(t) = \text{tr } B(t)$ . It remains to check (ii). A direct computation yields

$$\begin{aligned} \text{tr } \tilde{A}(t) &= a_{11}^2(t) + 2a_{12}(t)a_{21}(t) + a_{22}^2(t) + ca_{12}(t)[a_{11}(t) + a_{22}(t)] \\ &= [\text{tr } A(t)] \cdot [\text{tr } A(t) + ca_{12}(t)] - 2. \end{aligned}$$

Since  $\text{tr } A(t)$  is a non-constant polynomial, the whole expression is a non-constant polynomial for all sufficiently small  $c$  and we are done.  $\square$

Note that by the construction, the matrix  $V(t)$  is unimodular. Hence  $\tilde{S}(t)$  is unimodular if  $S(t)$  is such. This shows that in further constructions, based on Lemma 7, we may assume that the obtained matrix functions  $S_n(t)$  are unimodular.

In the following lemma, which can be regarded as a modification of Lemma 7, we preserve the notations  $\tilde{E}(t)$  and  $\tilde{S}(t)$ . For brevity, let us call a polynomial matrix function  $P(t) = (p_{ij}(t))$  *upper right dominating* if the degree of the polynomial  $p_{12}(t)$  is more than the degrees of others  $p_{ij}(t)$ .

**Lemma 8.** *Assume that  $(A(t), E)$  is a good pair and that the polynomial matrix  $A(t)^2$  is upper right dominating. Let  $\varepsilon > 0$  and  $N > 0$  be given. Then there exists  $\delta > 0$  and a compact interval  $I \subset (N, \infty)$  such that, for every real  $c$  with  $0 < |c| \leq \delta$ , there exists a compact set  $\tilde{E} \subset E$  such that  $|E \setminus \tilde{E}| < \varepsilon$ ;  $(\tilde{A}(t), \tilde{E} \cup I)$  is a good pair; and, moreover,  $\|\tilde{S}(t) - S(t)\| < \varepsilon$  on  $\tilde{E}$ .*

*Proof.* By the proof of Lemma 7, we get  $\delta_1$  such that for  $|c| < \delta_1$  one can find  $\tilde{E} \subset E$  satisfying the conditions:  $|E \setminus \tilde{E}| < \varepsilon$ ,  $(\tilde{A}(t), \tilde{E})$  is a good pair and  $\|\tilde{S}(t) - S(t)\| < \varepsilon$  on  $\tilde{E}$ .

Let  $A(t)^2 = (b_{ij}(t))$ . Since  $\tilde{A}(t) = A(t)M(c)A(t)$ ,

$$\text{tr } \tilde{A}(t) = \text{tr } [M(c)A(t)^2] = cb_{12}(t) + q(t),$$

where the degree of  $q(t)$  is less than the degree of  $b_{12}(t)$ . Hence if  $|c|$  is less than some  $\delta_2$  and has the appropriate sign, then there is  $t_0 > N$  for which  $\text{tr } \tilde{A}(t_0) = 0$ . So the condition (ii) holds on some interval  $I$  around  $t_0$ . This shows that  $(\tilde{A}(t), \tilde{E} \cup I)$  is a good pair.

It remains to set  $\delta = \min\{\delta_1, \delta_2\}$ .  $\square$

The number  $\delta$ , constructed in Lemma 8, will be denoted by  $\delta(A(t), E, \varepsilon, N)$ . To underline that in the construction of the interval  $I$  and the set  $\Omega = \widetilde{E} \cup I$  the number  $c$  from  $(-\delta, 0)$  or  $(0, \delta)$  is used, we will denote them by  $I = I(A(t), E, \varepsilon, N, c)$  and  $\Omega(A(t), E, \varepsilon, N, c)$  respectively.

Now we can prove the theorem.

**Proof of Theorem 3.** We shall start with the matrix  $A_0(t) = H(t)M_0H(t)$  and note that if  $c_0 < 0$ , then there exists a closed interval  $E_0 \subset (0, +\infty)$  of positive length such that  $(A_0(t), E_0)$  is a good pair. Choose  $\varepsilon_0 = |E_0|/3$ .

We will construct the sequences of numbers  $c_j, \varepsilon_j$ , matrix functions  $A_j(t)$ , compact sets  $E_j$  and compact intervals  $I_j$  inductively.

Suppose that these sequences are constructed for  $j < n$ . Then set

$$\varepsilon_n = \frac{1}{3^n} \min_{j < n} \{|I_j|\}, \quad \delta = \delta(A_{n-1}, E_{n-1}, \varepsilon_n, n)$$

and choose  $c_n$  with  $|c_n| < \delta$  and with appropriate sign. Let

$$I = I(A_{n-1}, E_{n-1}, \varepsilon_n, n, c_n), \quad E_n = \Omega(A_{n-1}, E_{n-1}, \varepsilon_n, n, c_n).$$

For these definitions be correct, we have to check that the pairs  $(A_n(t), E_n)$  are good and that  $A_n(t)^2$  are right upper dominating.

The first property follows by induction from Lemma 8. To prove the second one, note that for each  $n$ , the function  $A_n(t)^2$  is a product  $HMHMH \dots MH$  of matrices in which each  $H$  is either  $H(t)$  or  $H(2t)$  and each  $M$  is  $M(c)$  with some  $c \neq 0$  (possibly different for different  $M$ 's). Let  $p$  be the number of  $M$ 's in the product. Then  $A_n(t)^2 = (b_{ij}(t))$  where  $b_{ij}$  are polynomials with degrees of  $b_{11}(t)$  and  $b_{22}(t)$  equal  $p$ , degrees of  $b_{12}$  and  $b_{21}$  equal  $(p+1)$  and  $(p-1)$  respectively. The correctness is proved.

Let us set

$$E = \bigcup_{k=0}^{\infty} \bigcap_{n=k}^{\infty} E_n.$$

It follows from the choice of the numbers  $\varepsilon_j$  that  $|E \cap I_n| \neq 0$  for each  $n$ . Hence  $E$  is essentially unbounded. We have to prove that for each  $t \in E$  the

sequence  $\left\| \prod_{1 \leq k \leq K} \Phi_k H(t) \right\|$  is bounded.

If  $t \in E$  then there is  $m$  such that  $t \in E_n$  for each  $n \geq m$ . Then for any  $K$  which is divided by  $2^{m+1}$ , the partial product  $\prod_{1 \leq k \leq K} \Phi_k H(t)$  can be



written (see (21)) as

$$\prod_{1 \leq k \leq K}^{\widehat{\phantom{}}} \Phi_k H(t) = \prod_{1 \leq \ell \leq L}^{\widehat{\phantom{}}} M_{j_\ell} A_{j_\ell-1}(t) = \prod_{1 \leq \ell \leq L}^{\widehat{\phantom{}}} [M_{j_\ell} S_{j_\ell-1}(t) D_{j_\ell-1}(t) S_{j_\ell-1}(t)^{-1}]$$

where  $\{j_\ell\}$  is the strictly increasing finite sequence of integers such that  $K = 2^{j_1} + 2^{j_2} + \dots + 2^{j_L}$  and all  $j_\ell > m$ . To estimate the norm of this product, note that it consists of several diagonal matrices of norm 1, the matrix  $M_{j_1} S_{j_1-1}(t)$  in the beginning, the matrix  $S_{j_L-1}(t)^{-1}$  in the end and several matrices of the kind  $S_{j_\ell-1}(t)^{-1} M_{j_{\ell+1}} S_{j_{\ell+1}-1}(t)$  in the middle. Now,  $\|M_j\|$  are bounded. Also

$$\begin{aligned} \|S_j(t)\| &\leq \|S_j(t) - S_{j-1}(t)\| + \dots + \|S_{m+1}(t) - S_m(t)\| + \\ &\quad \|S_m(t)\| \leq \|S_m(t)\| + \sum_{j \geq m+1} \varepsilon_j \end{aligned}$$

are bounded for each such  $t$ . Since the matrices  $S_j(t)$  are unimodular, their inverse are also bounded:

$$\|S_j(t)\| \leq C(t), \quad \|S_j(t)^{-1}\| < C(t).$$

It remains to estimate the norms of  $S_{j_\ell-1}(t)^{-1} M_{j_{\ell+1}} S_{j_{\ell+1}-1}(t)$ . We have

$$\begin{aligned} \|S_{j_\ell-1}(t)^{-1} M_{j_{\ell+1}} S_{j_{\ell+1}-1}(t) - I\| &= \\ \|S_{j_\ell-1}(t)^{-1} \left( (M_{j_{\ell+1}} - I) S_{j_{\ell+1}-1}(t) + (S_{j_{\ell+1}-1}(t) - S_{j_\ell-1}(t)) \right)\| &\leq \\ \|S_{j_\ell-1}(t)^{-1}\| \cdot \left( \|M_{j_{\ell+1}} - I\| \cdot \|S_{j_{\ell+1}-1}(t)\| + \|S_{j_{\ell+1}-1}(t) - S_{j_\ell-1}(t)\| \right) & \\ \leq C(t) \left( |c_{j_{\ell+1}}| C(t) + \sum_{j=j_\ell}^{j_{\ell+1}-1} \varepsilon_j \right). \end{aligned}$$

Hence

$$\|S_{j_\ell-1}(t)^{-1} M_{j_{\ell+1}} S_{j_{\ell+1}-1}(t)\| \leq \exp \left\{ C(t) \left[ C(t) |c_{j_{\ell+1}}| + \sum_{j=j_\ell}^{j_{\ell+1}-1} \varepsilon_j \right] \right\}.$$

Multiplying all the above estimates, we see that, for  $K \in 2^{m+1}\mathbb{Z}$ , one has

$$\left\| \prod_{1 \leq k \leq K}^{\widehat{\phantom{}}} \Phi_k H(t) \right\| \leq C^2(t) \exp \left\{ C(t) \left[ C(t) \sum_{j \geq 1} |c_j| + \sum_{j \geq 1} \varepsilon_j \right] \right\}.$$

Therefore the partial products corresponding to  $K \in 2^{m+1}\mathbb{Z}$  are bounded for each  $t \in \bigcap_{n \geq m} E_n$ . The products corresponding to other  $K$  differ from the products corresponding to  $K \in 2^{m+1}\mathbb{Z}$  by just  $N$  couples of (uniformly) bounded matrices ( $N \in \{1, 2, \dots, 2^{m+1} - 1\}$ ). Therefore, all the sequence of partial products is bounded for such  $t$ .

□

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